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ORIGINAL PAPER

Interplay between interior and boundary geometry in Gromov hyperbolic spaces

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Abstract We show that two visual and geodesic Gromov hyperbolic metric spaces are roughly isometric if and only if their boundaries at infinity, equipped with suitable quasimetrics, are bilipschitz-quasimoebius equivalent. Similarly, they are quasi-isometric if and only if their boundaries are power quasimoebius equivalent.

Keywords Hyperbolic spaces · Boundary at infinity · Quasimetric · Quasisymmetric maps · Quasimoebius maps

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1 Introduction

Given a Gromov hyperbolic metric space X, one has associated to it the boundary at infinity, or ideal boundary, $\partial_\infty X$. Via the Gromov product $(\cdot|\cdot)$, one obtains in canonical fashion a family of quasimetrics $a^{-(\cdot|\cdot)}$ on the set $\partial_\infty X$. It is a very natural question to ask to what extent the structure of the boundary determines the space itself, and what kind of correspondence exists between maps of spaces and maps between their associated boundaries. Previous results in this direction were obtained by Paulin [6], Bonk and Schramm [1], and Buyalo and Schroeder [3], among others. These all differ somewhat among one another in the class of spaces and maps they are valid for. The goal of this work was to find a general setting that systematically explores the relationship of a Gromov hyperbolic space to its boundary and vice versa. We are then able to deduce the cited results as special cases within this general framework, cf. Corollaries 4 and 5.

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At the heart of this work lie the extension theorems for bilipschitz, power quasisymmetric and power quasimoebius boundary maps, which we can subsume in the

Theorem 1 (Cf. Theorems 4, 5, 8) *Let X, X' be hyperbolic metric spaces with X visual and X' roughly geodesic.*

If $f : \partial_\infty^{a,o} X \to \partial_\infty^{a,o'} X'$ is a bilipschitz map, then there exists a rough isometric map $F : X \to X'$ such that its naturally associated boundary map $\partial_\infty F$ equals f, $\partial_\infty F = f$.

If $f : \partial_\infty^{o} X \to \partial_\infty^{o'} X'$ is a power quasisymmetric map, then there exists a power quasi-isometric map $F : X \to X'$ such that $\partial_\infty F = f$.

If $f : \partial_\infty X \to \partial_\infty X'$ is a power quasimoebius map, then there exists a power quasi-isometric map $F : X \to X'$ such that $\partial_\infty F = f$.

See Definitions 7, 8 and its subsequent paragraph for the definition of power quasi-isometric and quasimoebius/quasisymmetric maps.

For spaces which are both visual and roughly geodesic one then obtains the following characterization of rough isometry and PQ-isometry classes.

Theorem 2 *Let X, X' be visual, roughly geodesic hyperbolic metric spaces. The following are mutually equivalent.*

(I) *X and X' are roughly isometric.*
(II) *There is a map $F : X \to X'$ and a $D \geq 0$ such that for all quadruples $Q \subset X$*

$$\mathrm{cd}(Q) - D \leq \mathrm{cd}(F(Q)) \leq \mathrm{cd}(Q) + D.$$

(III) *For any $a > 1$ there is a bilipschitz-quasimoebius homeomorphism $f : \partial_\infty^a X \to \partial_\infty^a X'$.*

Also the following are equivalent.

(i) *X and X' are quasi-isometric.*
(ii) *X and X' are power quasi-isometric.*
(iii) *For any $a, a' > 1$, $\partial_\infty^a X$ is power quasimoebius equivalent to $\partial_\infty^{a'} X'$.*

Note $(II) \Rightarrow (I)$ and $(ii) \Rightarrow (i)$ are trivial, as is $(I) \Rightarrow (II)$. The implication $(i) \Rightarrow (ii)$ is due to Buyalo and Schroeder ([3], Theorem 4.4.1). The bilipschitz and the power quasi-symmetric extension theorems were proved in the metric setting by Bonk and Schramm in [1], Theorem 7.4.

The main contributions of this paper are the quasimetric extension theorems for power quasisymmetric maps (Theorem 5) and inversions (Theorem 7), which are combined to give the extension for power quasimoebius maps (Theorem 8).

This article is organized as follows. Section 2 recalls basic notions on Gromov hyperbolic spaces and gives definitions on quasimetric spaces and the various classes of morphisms between (quasi) metric spaces we consider in this article. Section 3 summarizes the technique of producing a Gromov hyperbolic space to a given boundary via hyperbolic approximation. Section 4 recalls the well-known theorem on extension of bilipschitz boundary maps, while Sects. 5 and 6 contain the proofs for the extension theorems for power quasisymmetric and inversion maps, respectively. Section 7 combines them to prove the the extension theorem for power quasimoebius maps. Section 8 combines the pieces to prove Theorems 1 and 2.

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2 Preliminaries and notation

2.1 Some notation

The notation $a \asymp_K b$ is shorthand for $a/K \leq b \leq Ka$, $a \doteq_C b$ stands for $a-C \leq b \leq a+C$. For example, saying that $|F(x)F(y)| \asymp_K |xy|\ \forall x, y \in X$, or $|F(x)F(y)| \doteq_C |xy|\ \forall x, y \in X$ is another way of saying that the map $F : X \to Y$ is bilipschitz or roughly isometric, respectively. If we do not specify the constants K or C and just write $a \asymp b$, $a \doteq b$, it is understood that there is a *uniform* such constant which works for all a and b in the given context.

At some point we will also use $a \dot{\geq} b$, which will analogously mean that there is a uniform C such that $a \geq b - C$.

For metric spaces we find it convenient to denote the metric by $|\cdot|$, that is the distance from x to y is written as $|xy|$.

2.2 Gromov hyperbolic spaces

Definition 1 Given $\delta \geq 0$ and $T = (x_0, x_1, x_2)$ a triple of real numbers, we say that T is a δ*-triple* if the two smaller numbers differ by no more than δ, or equivalently if the δ*-inequality*

$$x_i \geq \min\{x_{i-1}, x_{i+1}\} - \delta \quad \forall i \in \mathbb{Z}/3\mathbb{Z}$$

is satisfied.

Definition 2 Let $(X, |\cdot|)$ be a metric space and $x, y, o \in X$ The *Gromov product of* x *and* y *with respect to* o, $(x|y)_o$ is defined as

$$(x|y)_o := \frac{1}{2}(|ox| + |oy| - |xy|).$$

$(X, |\cdot|)$ is called δ*-hyperbolic* if for all $x, y, z, o \in X$ the triple

$$((x|y)_o, (x|z)_o, (y|z)_o)$$

is a δ-triple.

$(X, |\cdot|)$ is called *Gromov hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

A *geodesic* in a metric space X is an isometric map $\gamma : I \to X$ from a real interval (possibly infinite) into X. X is called *geodesic* if for any two points $x, y \in X$ there exists a geodesic $\gamma : [a, b] \to X$ with $\gamma(a) = x$ and $\gamma(b) = y$.

2.3 Boundary at infinity

A sequence (x_i) in a metric space is said to *converge to infinity* if

$$\lim_{i,j\to\infty} (x_i|x_j)_o = \infty$$

for one, and hence any, base point $o \in X$.

Two sequences (x_i), (x_i') are said to be equivalent if

$$\lim_{i\to\infty} (x_i|x_i')_o = \infty.$$

For Gromov hyperbolic spaces, this defines an equivalence relation and we define the set called *boundary at infinity of* X, $\partial_\infty X$, to be the set of equivalence classes of sequences converging to infinity.

Let $\xi, \xi' \in \partial_\infty X$ and $o \in X$. We extend the Gromov product to the boundary at infinity by setting

$$(\xi|\xi')_o := \inf \liminf_{i\to\infty} (x_i|x_i')_o,$$

where the infimum is taken over all sequences $(x_i) \in \xi$, $(x_i') \in \xi'$. It is a fact ([3], Lemma 2.2.2(2)) that with this definition, the δ-inequality extends to the boundary at infinity. That is,

$$((\xi|\xi')_o, (\xi|\xi'')_o, (\xi'|\xi'')_o)$$

is a δ-triple for all $\xi, \xi', \xi'' \in \partial_\infty X$.

2.4 Busemann functions

With the help of the Gromov product for boundary points defined above we can give the set $\partial_\infty X$ the structure of a *bounded* quasimetric space, see below. To get nonbounded (quasi)metrics on the boundary, however, we need to introduce Busemann functions.

For $\omega \in \partial_\infty X$ and $o \in X$ define the function

$$b_{\omega,o} : X \to \mathbb{R}$$
$$x \mapsto b_{\omega,o}(x) := (\omega|o)_x - (\omega|x)_o,$$

where the Gromov product $(\omega|o)_x$, with one argument in $\partial_\infty X$, is defined as $\inf \liminf (w_i|o)_x$ in analogy to the case with both arguments in $\partial_\infty X$.

$b_{\omega,o}$ is the prototype of a Busemann function based at $\omega \in \partial_\infty X$. It corresponds to the Busemann function associated to a geodesic ray from o to ω in case X is a Riemannian manifold of pinched negative curvature. Any function that is equal to $b_{\omega,o}$ up to a constant and a uniformly bounded additive error shall be called a *Busemann function*. More precisely:

Definition 3 Let $\omega \in \partial_\infty X$. The set $\mathcal{B}(\omega)$ of all *Busemann functions based at* ω consists of all those functions $b : X \to \mathbb{R}$ for which there exists $o \in X$ and a constant $c \in \mathbb{R}$ such that $b \doteq_{2\delta} b_{\omega,o} + c$.

For $b \in \mathcal{B}(\omega)$ a Busemann function based at ω, we define the Gromov product $(x|y)_b$ for $x, y \in (X, |\cdot|)$ by

$$(x|y)_b := \frac{1}{2}(b(x) + b(y) - |xy|).$$

Note that $(\cdot|\cdot)_b$, in contrast to $(\cdot|\cdot)_o$, can be negative.

We extend the Gromov product to $\partial_\infty X$ by

$$(\xi|\xi')_b := \inf \liminf_{i\to\infty} (x_i|x_i')_b,$$

where the infimum is taken over all sequences $(x_i) \in \xi$, $(x_i') \in \xi'$.

Proposition 1 ([3], Lemma. 3.2.4(2)) *For X a δ-hyperbolic space and $\xi, \eta, \zeta, \omega \in \partial_\infty X$ arbitrary, the numbers $(\xi|\eta)_b, (\xi|\zeta)_b, (\eta|\zeta)_b$ form a 22δ-triple for any $b \in \mathcal{B}(\omega)$.*

2.5 Quasimetric spaces

Definition 4 A K*-quasimetric space* is a set Z together with a map $\rho : Z \times Z \to [0, \infty]$ such that

I. $\rho(z, y) \geq 0 \; \forall z, y \in Z$, with equality iff $y = z$.
II. $\rho(z, y) = \rho(y, z) \forall z, y \in Z$.
III. $\rho(z, w) \leq K \max\{\rho(z, y), \rho(y, w)\} \; \forall w, y, z \in Z$.
IV. There is at most one $z \in Z$ such that $\rho(z, y) = \infty$ for all $y \in Z \backslash \{z\}$.

If no point z as in *IV* exists, Z is said to be *non-extended*, while it is *extended* if there is such a z and this z is then called the *infinitely remote point*. By convention, a one-point space $Z = \{z\}$ is never extended.

If X is a δ-hyperbolic space, $a > 1$, $o \in X$ and $(\cdot|\cdot)_o$ denotes the Gromov product with respect to the base point o, then $a^{-(\cdot|\cdot)_o}$ is an a^{δ}-quasimetric on the set $\partial_{\infty} X$. Similarly, $a^{-(\cdot|\cdot)_b}$, for some Busemann function b, defines an $a^{22\delta}$-quasimetric.

In particular, the boundary at infinity of a 0-hyperbolic space (i.e. a subset of a tree) is K-quasimetric with $K = 1$. Such spaces are usually called ultrametric spaces.

A quasimetric ρ on a space Z induces a topology by declaring a set $A \subset Z$ to be open if for every $a \in A \backslash \{\infty\}$ there exists $r > 0$ such that $B_r^{\rho}(a) \subset A$, and if $\infty \in A$, then there exists $y \in Z$ and $r > 0$ such that $A \subset B_r(y)^c$. This topology is metrizable and in particular first-countable and Hausdorff. This follows from the fact that if (Z, ρ) is K-quasimetric, then (Z, ρ^s) is K^s-quasimetric (and the two topologies are clearly equivalent), and a result of Frink's ([4]) whereby a K-quasimetric with $1 \leq K \leq 2$ is bilipschitz equivalent to a metric (extended if ρ is extended).

Here and in the future we always denote $B_r^{\rho}(x) := \{z \in Z | \rho(z, x) < r\}$. Note, though, that in contrast to the metric setting this need not be an open set.

Definition 5 A quasimetric space (Z, ρ) is called *complete* if every Cauchy sequence in $Z \backslash \{\infty\}$ converges and if ρ is extended in case it is unbounded.

For example, the circle S^1 with the induced metric from $\mathbb{R}^2$ is complete. $\mathbb{R}$ is not complete but $\mathbb{R} \cup \infty$ is. Note that $\mathbb{R} \cup \infty$ is obtained from S^1 via stereographic projection, a Moebius map. It is true in general that quasimoebius maps (see below) send complete spaces to complete spaces.

Boundaries of hyperbolic spaces are always complete, cf. [1], Proposition 6.2.

Definition 6 The symbol $\partial_{\infty} X$ denotes the *set* of boundary points of a Gromov hyperbolic space. The symbols $\partial_{\infty}^{a,o}$ and $\partial_{\infty}^{a,b}$, where $a > 1$, $o \in X$, $b \in \mathcal{B}(\omega)$, denote the *quasimetric spaces* $(\partial_{\infty} X, a^{-(\cdot|\cdot)_o})$ and $(\partial_{\infty} X, a^{-(\cdot|\cdot)_b})$, respectively.

Remark 1 In fact, the bilipschitz class of $\partial_{\infty}^{a,o} X$ does not depend on $o \in X$ and the quasimoebius class depends on neither of the parameters. Thus we may suppress one or both of them and just write $\partial_{\infty}^{a} X$, or $\partial_{\infty} X$. Whenever we do this it is to be understood that the statement holds for any admissible choice of the omitted parameter(s).

Note that $\partial_{\infty}^{o} X$ is always bounded, while $\partial_{\infty}^{b} X$, for $b \in \mathcal{B}(\omega)$, is always extended with infinitely remote point ω.

2.6 Various classes of maps

A map $f : X \to Y$ between metric spaces is called *roughly isometric*, or more specifically C*-roughly isometric* if there exists C such that $|xy| \doteq_C |f(x) f(y)|$ for all $x, y \in X$. f is called *quasi-isometric*, or (c, d)-*quasi-isometric*, if there exist c, d such that $\frac{1}{c}|xy| - d \leq |f(x)f(y)| \leq c|xy| + d$. If there exists a roughly isometric (quasi-isometric) map $g : Y \to X$ such that $d_X(g \circ f(x), x) \leq D$ for some uniform D, then f is called a rough isometry (quasi-isometry) and X is said to be *roughly isometric (quasi-isometric) to* Y.

A metric space X is called *roughly geodesic* if there exists for any $x, y \in X$ a C-*rough geodesic* joining x and y, where a C-rough geodesic is a C-roughly isometric map from an interval $I \subset \mathbb{R}$ into X.

If a map $F : X \to X'$ between Gromov hyperbolic spaces maps sequences going to infinity in X to sequences going to infinity in X' and equivalent sequences to equivalent sequences, then F induces a map between boundaries, which we denote $\partial_\infty F : \partial_\infty X \to \partial_\infty X'$.

For example, every roughly isometric map $F : X \to X'$ induces an injection $\partial_\infty F : \partial_\infty X \to \partial_\infty X'$. A quasi-isometric map $F : X \to X'$ between *geodesic* hyperbolic spaces induces a boundary map by the stability of geodesics (cf. [2], Theorem III.H.1.7). However, the map $F : \{10^i | i \in \mathbb{N}\} \to \mathbb{R}$, $F(10^i) := (-1)^i 10^i$ is quasi-isometric, but does not induce a boundary map in any reasonable sense. This is one of the reasons why quasi-isometric maps are in general not the right maps to look at in the setting of hyperbolic metric spaces. In fact, for non-geodesic spaces, quasi-isometric maps need not even preserve Gromov hyperbolicity. The following definition gives the class of maps with all desired properties.

Definition 7 For $Q = (x, y, z, w)$ an ordered quadruple of points in a metric space $(X, |\cdot|)$ denote by $cd(Q)$ their *cross-difference*,

$$cd(Q) := \frac{1}{2}(|xz| + |yw| - |xy| - |zw|) = (x|y)_o + (z|w)_o - (x|z)_o - (y|w)_o.$$

A map $F : X \to X'$ between metric spaces is called (c, d)-*power quasi*-isometric (PQ-isometric) if

$$\frac{1}{c} cd(Q) - d \leq cd(F(Q)) \leq c\, cd(Q) + d.$$

Since $cd(x, x, y, y) = |xy|$, every power quasi-isometric map is quasi-isometric. Moreover, every power quasi-isometric map $F : X \to X'$ between hyperbolic spaces induces a boundary map $\partial_\infty F : \partial_\infty X \to \partial_\infty X'$. This follows from $cd(x, y, o, o) = (x|y)_o$.

A quasi-isometric map between *geodesic* hyperbolic spaces is automatically PQ-isometric, cf. [3], Theorem 4.4.1 (what we call PQ-isometric is called *strongly* PQ-isometric in [3]).

The multiplicative analog of a PQ-isometric map is a *power quasimoebius map*.

Definition 8 For $Q = (x, y, z, w)$ an ordered quadruple of points in a quasimetric space (Z, ρ) denote by $cr(Q)$ their *cross-ratio*

$$cr(Q) = \frac{\rho(x, z)\rho(y, w)}{\rho(x, y)\rho(z, w)}.$$

If $\theta : [0, \infty) \to [0, \infty)$ is a homeomorphism, a map $f : Z \to Z'$ between quasimetric spaces is called θ-quasimoebius (θ-QM) if $1/\theta(cr(Q)^{-1}) \leq cr(f(Q)) \leq \theta(cr(Q))$. f is called *power quasimoebius* (P-QM) if it is θ-QM for a θ of the form $\theta(t) = q \max\{t^{1/p}, t^p\}$. It is called *bilipschitz quasimoebius* if θ can be taken of the form $\theta(t) = \lambda t$.

Closely related to QM maps are *quasisymmetric* (QS) maps, which are the ones which preserve the ordinary ratio sr of a triple (x, y, z), $sr(x, y, z) := |xz|/|xy|$, in an analogous way.

We refer to [10] and [3], Chapter 5, for more information on quasimoebius and quasisymmetric maps. Quasimoebius maps are called “strictly quasimoebius” in [3].

3 Hyperbolic approximation

The goal of hyperbolic approximation is to find to a given (quasi)metric space Z a hyperbolic space X (with nice properties) such that $\partial_\infty X = Z$. The procedure we use was developed by Buyalo and Schroeder, cf. Chapter 6 of [3]. However, the idea of constructing a hyperbolic space with prescribed boundary is itself not new. The usual approach has been to mimic the upper half plane or the unit disk situation by crossing the given space with $\mathbb{R}_{\geq 0}$ (or a finite interval in the case of bounded boundary) and equipping the product with a suitable metric which turns out to be hyperbolic. The oldest such method may be the hyperbolic cone over a metric case, cf. [3] §6.4.4, originally due to Berestovskii. Similar constructions were also used by Gromov, Trotsenko and Väisälä (cf. [8]), as well as Bonk and Schramm (cf. [1]). Buyalo and Schroeder's method has the advantage that it is very intuitive and produces a particularly nice geodesic space, namely a graph, which is easily recognized to be visual. Basically, only the verification of hyperbolicity needs some work. Furthermore, it is straightforward to adapt it to the setting of quasimetric boundary spaces, which is crucial for this work.

Let (Z, ρ) be a complete K-quasimetric space. Let $r < 1/K^3$. The procedure now goes as follows. For every $k \in \mathbb{Z}$ let V_k be a maximal r^k-separated subset of Z (such exist by Zorn), where r^k-separated means $\rho(v, v') \geq r^k$ for all $v, v' \in V_k$. Denote by $\mathcal{V}$ the set of all ordered pairs (k, z) with $k \in \mathbb{Z}$ and $z \in V_k$. The projection $\ell : \mathcal{V} \to \mathbb{Z}$ to the first coordinate is called *level function*, and $\ell(v)$ the *level* of v, while the projection $\pi : \mathcal{V} \to Z$ to the second coordinate sends v to its *center* $\pi(v) \in Z$.

Remark 2 Sometimes the notation $\pi(v)$ becomes too cumbersome so that we often identify a pair $v \in V_k$ with its center $\pi(v) \in Z$. The notation $\rho(v, w)$ is thus interpreted to mean $\rho(\pi(v), \pi(w))$.

The hyperbolic approximation with parameter $r < 1/K^3$ is then defined to be the simplicial graph with vertex set $\mathcal{V}$, where two vertices $v, w \in \mathcal{V}$ are joined by an edge exactly when

- $\ell(v) = \ell(w)$ and the sets $B(v) := B_{Kr^{l(v)}}(\pi(v))$ and $B(w) := B_{Kr^{l(w)}}(\pi(w))$ intersect in Z, or
- $\ell(v) = \ell(w) + 1$ and $B(v)$ is contained in $B(w)$.

It follows from [3], Theorems. 6.3.1, 6.4.1, (cf. Theorems. 3 below) that then $\partial_\infty^{1/r}\mathrm{Hyp}_r$ (Z, ρ) is bilipschitz equivalent to (Z, ρ). So far this only holds for $r < 1/K^3$. Now the boundaries at infinity come equipped with a family of quasimetrics $a^{-(\cdot|\cdot)}$ for $a > 1$. The corresponding situation for hyperbolic approximations is that they should be taken for a family of parameters $r \in (0, 1)$, not just for $r \in (0, 1/K^3)$. Even though it should intuitively be possible to make a similar construction with balls as above, it seems the resulting graph is too difficult to control. For this reason, we resort to a scaling trick.

Definition 9 Let (Z, ρ) be a complete K-quasimetric space and $r \in (0, 1)$. If Z is extended and $|Z| \geq 3$ with ξ the infinitely remote point, define $\mathrm{Hyp}_r(Z, \rho)$ to be the graph obtained from $(Z \backslash \{\xi\}, \rho^{1/s})$ as above when $r < 1/K^3$, and define it to be the graph obtained for an $r' < 1/K^3$ scaled by $\frac{\ln r'}{\ln r}$ when $r \geq 1/K^3$. If $|Z| = 2$ define $\mathrm{Hyp}_r(Z) := \mathbb{R}$.

If (Z, ρ) is not extended and hence bounded, then for $|Z| \geq 2$, $\mathrm{Hyp}_r(Z, \rho)$ is defined in the same way except that it is understood to be truncated (cf. [3] §6.4.1). For $|Z| = 1$ define $\mathrm{Hyp}_r(Z) := \mathbb{R}_{\geq 0}$.

It is not difficult to show that, up to a rough isometry, the resulting graph does not depend on the choice of vertex system $\mathcal{V}$, nor on the quasimetricity constant K used for ρ (note a K-quasimetric is also a K'-quasimetric for $K' \geq K$). Moreover, the rough isometry class of $\mathrm{Hyp}_r(Z, \rho)$ does not depend on the choice of r' in Definition 9, meaning one has $\frac{\ln r_1}{\ln r_2} \mathrm{Hyp}_{r_1}(Z, \rho) \doteq \mathrm{Hyp}_{r_2}(Z, \rho)$. This can be proved directly with Lemma 7 although for *bounded* ρ it also follows from the bilipschitz extension Theorem 4.

We remark that by a Zorn-type argument there exist *hereditary* vertex systems $\mathcal{V} = \{V_k\}_k$, meaning that $\pi(V_k) \subset \pi(V_{k+1})$. Working with such hereditary systems often simplifies arguments and we will use them without reservation when it suits us.

In the extended case, $\mathrm{Hyp}(Z)$ has a distinguished boundary point ω corresponding to the infinitely remote point ξ of Z, while in the non-extended case the root o of the approximation will serve as distinguished base point.

The crucial theorem about hyperbolic approximation is

Theorem 3 ([3], Theorems 6.3.1, 6.4.1) *Let (Z, ρ) be a complete quasimetric space, $r \in (0, 1)$. The hyperbolic approximation $\mathrm{Hyp}_r(Z)$ is a visual geodesic hyperbolic space and there is a canonical identification $\partial_\infty \mathrm{Hyp}_r(Z) = Z$ of sets. Moreover, if (Z, ρ) is extended then for any $b \in \mathcal{B}(\omega)$, $\partial_\infty^{1/r, b} \mathrm{Hyp}_r(Z, \rho)$ and (Z, ρ) are bilipschitz equivalent. If (Z, ρ) is not extended, then $\partial_\infty^{1/r, o} \mathrm{Hyp}_r(Z, \rho)$ and (Z, ρ) are bilipschitz equivalent.*

The moral of the story is that, given a complete quasimetric space (Z, ρ), there is for every $a > 1$ exactly one (up to rough isometry) visual geodesic hyperbolic space X such that $\partial_\infty^a X$ is bilipschitz-quasimoebius to (Z, ρ), and the "functor" $\mathrm{Hyp}_{1/a}$ spits out exactly this space X when applied to (Z, ρ).

4 Extension of bilipschitz maps

We recall [3], Theorem 7.1.2, stated here for quasimetric boundary spaces. The proof is exactly the same as in the metric setting of [3].

Theorem 4 *Let X be a visual and X' be a geodesic hyperbolic space, $o \in X, o' \in X'$. Then to every bilipschitz map $f : \partial_\infty^{a,o}, X \to \partial_\infty^{a,o'} X'$, there exists a roughly isometric map $F : X \to X'$ with $\partial_\infty F = f$.*

Corollary 1 ([3], Corollaries 7.1.5, 7.1.6 and [1], Theorem 8.2) *Let X be a visual hyperbolic space and $o \in X$, $a > 1$, $r \in (0, 1)$.*

X embeds roughly homothetically into $\mathrm{Hyp}_r \partial_\infty^{a,o} X$. If X is also roughly geodesic, then there is a rough homothety of X onto $\mathrm{Hyp}_r \partial_\infty^{a,o} X$.

In addition, X embeds roughly isometrically into $\mathrm{Hyp}_{1/a} \partial_\infty^{a,o} X$. If X is also roughly geodesic, then X is roughly isometric to $\mathrm{Hyp}_{1/a} \partial_\infty^{a,o} X$.

When we are only concerned about the quasi-isometry class of the approximation, it is thus not necessary to specify the parameter r in $\mathrm{Hyp}_r(Z)$. Whenever we write only $\mathrm{Hyp}(Z)$ in a statement, it is to be understood that the statement is true for *every* $r \in (0, 1)$.

5 Extension of PQ-symmetric maps

In this section we prove

Theorem 5 (Compare [1], Theorem 7.4) *Let* $(Z, \rho), (Z', \rho')$ *be two bounded complete quasi-metric spaces and* $\mathrm{Hyp}(Z), \mathrm{Hyp}(Z')$ *be their hyperbolic approximations. Suppose* $f : Z \to Z'$ *is a power quasisymmetric homeomorphism, i.e.* η*-QS with* $\eta(t) = C \max\{t^\alpha, t^{1/\alpha}\}$ *for some* $C > 0, \alpha \geq 1$. *Then there exists a power quasi-isometry* $F : \mathrm{Hyp}(Z) \to \mathrm{Hyp}(Z')$ *with* $\partial_\infty F = f$.

This theorem is trivial for $Z = \{z\}$, so we assume $|Z| \geq 2$. For convenience we shall also assume throughout this section that both spaces are K-quasimetric and that the approximations of both spaces are done w.r.t the *same* parameter $r = 1/(2K^3)$. This poses no loss of generality by Theorem 3, Corollary 1 and indepence of K of the hyperbolic approximation.

We assume the vertex system $\mathcal{V} = \{V_k\}$ is hereditary. We will split up the vertices into two disjoint subsets. Recall that if $v \in V_k$, then to v is associated the ball $B(v) = B_k(v) := B_{Kr^k}(\pi(v)) \subset Z$.

Definition 10 A vertex $v \in V_k$ is called *regular* if the annulus $B_{Kr^k}(\pi(v)) \backslash B_{Kr^{k+1}}(\pi(v))$ is non-empty. It is called *singular* if it is not regular.

The root o of a truncated hyperbolic approximation is always regular unless $Z = \{z\}$, which we assume is not the case.

Lemma 1 *If* $v \in V_k$ *is singular and connected radially to a vertex* $w \in V_{k+1}$ *and* $\pi(w) \neq \pi(v)$*, then* w *is regular and so is* $(\pi(v), k+1) \in V_{k+1}$.

Moreover, if w *is a horizontal neighbour of* $v \in V_k$*, then at least one of* v, w *is regular.*

Proof $B(w) \subset B(v)$ by definition of radial edges. Since v is singular, this means $B(w) \subset B_{Kr^{k+1}}(\pi(v))$. On the other hand, $\rho(\pi(v), \pi(w)) \geq r^{k+1} > Kr^{k+2}$, which means w and $(\pi(v), k+1)$ are regular.

If $v, w \in V_k$ are both singular, then $\rho(v, w) \geq r^k$ and for any $z \in B(v)$, $\rho(v, z) < Kr^{k+1}$ by singularity of v. Hence $\rho(w, z) \geq r^k/K > Kr^{k+1}$ and z is not in $B(w)$, hence $B(v) \cap B(w) = \emptyset$. □

Remark 3 If $v \in V_k$ and $B_{Kr^{k-1}}(v) \supsetneq B_{Kr^k}(v)$, $(\pi(v), k-1)$ may or may not be in V_{k-1}. At any rate, we know by maximality of V_{k-1} that there exists $w \in V_{k-1}$ which is radially connected to $v \in V_k$ and $\rho(\pi(w), \pi(v)) < r^{k-1}$.

We will now define the map F of Theorem 5. The idea is to define it first on all regular vertices and then "fill in" the rest. First of all note the

Lemma 2 *For any vertex* $v \in V_k$ *of a hereditary vertex system* $\mathcal{V}$ *exactly one of the following holds.*

I. v *is regular*
II. v *is singular and so are* $v \in V_{k+l}$ *for* $0 \leq l < N$*, while* $v \in V_{k+N}$ *is regular.* $N \geq 1$.
III. v *is singular in* V_{k+l} *for all* $l \geq 0$.

Proof The notation $v \in V_{k+l}$ is meant to denote the element $(\pi(v), k+l)$ of V_{k+l}. The cases are mutually exclusive and exhaustive, so the lemma is evident. □

We will refer to the numbers in Lemma 2 as the *types* of a given vertex $v \in V_k$, type I vertices being the regular vertices and so on, cf. Fig. 1.

Definition 11 If $v \in \mathcal{V}$ is regular, $F(v)$ is defined to be a vertex $v' \in \mathrm{Hyp}(Z')$ of highest level such that $B(v') \supset f(B(v))$.

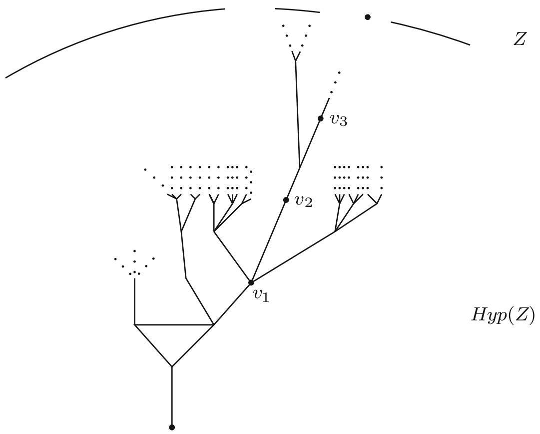

Fig. 1 v_1, v_2, v_3 are vertices of type I, II, III, respectively

This defines F on the set of regular vertices up to an error of at most 1, as any two such vertices v' are evidently connected by an edge.

Lemma 3 *If $v \in V_k$ is regular, then so is $F(v)$.*

Proof Denote by m the level $l(F(v))$ of $F(v)$ in $\mathrm{Hyp}(Z')$. If $F(v)$ were singular, $B_{K'r^m}(\pi(F(v)))\backslash B_{K'r^{m+1}}(\pi(F(v)))$ would have to be empty. This would mean that all of $f(B(v))$ would already be contained in $B_{K'r^{m+1}}(\pi(F(v)))$, contradicting the maximality of the level of $F(v)$ among all vertices containing $f(B(v))$. □

Now suppose $v \in V_k$ is of type II. As noted before, v is not the root of $\mathrm{Hyp}(Z)$. In particular, there will be an $m \in \mathbb{N}$ and a $w \in V_{k-m}$ such that w is regular, $v \in V_{k-m+1}$ and singular, and $w \in V_{k-m}$ is radially connected to $v \in V_{k-m+1}$. $\pi(w)$ may or may not be equal to $\pi(v)$, confirm Remark 3. Trivially, all the v's on adjacent levels are radially connected. We define the following terms.

Definition 12 A geodesic segment in $\mathrm{Hyp}(Z)$ through vertices $v_0, \ldots, v_N$ is called *singular* if the vertices v_1 up to and including v_{N-1} are all singular.

By Lemma 1 and the paragraph following this definition, we may assume that all edges $v_0v_1, \ldots, v_{N-1}v_N$ are radial and that $\pi(v_1) = \cdots = \pi(v_N)$. It follows that the level function is monotonous along the geodesic and, after possibly reversing the order, we may assume

$$k - m = \ell(v_0) \leq \ell(v_1) < \ell(v_2) \cdots < \ell(v_{N-1}) \leq \ell(v_N) = k + l.$$

If v_0 is regular, we call it the *lower end* of the singular geodesic and if v_N is regular, it is the *upper end*, respectively.

Every singular geodesic segment has a lower end since the root is regular. A singular geodesic with no upper end is called a *singular ray*. The lower end of a singular ray is also called its *root*.

Lower and, if they exist, upper ends are uniquely determined by the singular segment up to error 1. In particular, if $v_N \in V_N$ is an upper end we may assume $\pi(v_N) = \pi(v_{N-1})$ because if v_{N-1} is singular, then $(\pi(v_{N-1}), N)$ is connected to v_N and both are regular. Similarly, if v_0 is a lower end and $\ell(v_0) = \ell(v_1)$, then any $w \in V_{\ell(v_o)-1}$ with $\rho(w, v_1) < r^{\ell(v_0)-1}$ is regular

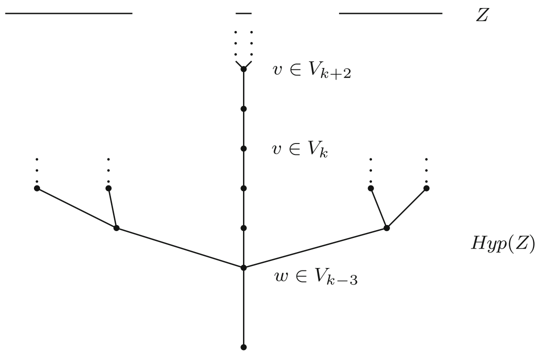

Fig. 2 Singular geodesic $w \dots v_{k+2}$ associated to $v_k = v \in V_k$

and we can replace v_0 with w. We may thus suppose $\ell(v_0) < \ell(v_1) < \cdots < \ell(v_N) = k + l$ With these assumptions, a vertex $v \in V_k$ of type II thus gives rise to a singular geodesic segment $wv_{k-m+1} \cdots v_k \cdots v_{k+l}$ with lower end $w \in V_{k-m}$ and upper end $v \in V_{k+l}$, where $\pi(v_{k-m+1}) = \cdots = \pi(v_{k+l})$ and every edge of which is radial. Cf. Fig. 2.

The hope is now that $F(w)$ and $F(v_{k+l})$ will be joined in $\mathrm{Hyp}(Z')$ by a singular segment whose length is in bilipschitz correspondence to $|wv_{k+l}| = m+l$. This turns out to be roughly true, cf. Lemmata 5 and 6.

Lemma 4 *Suppose $v_k \in V_k$ is of type II and $v_{k+l} \in V_{k+l}$, $w \in V_{k-m}$ are the upper and the lower ends of the singular geodesic associated to $v_k \in V_k$. There is a $C_1 = C_1(\eta, K, r)$ such that if $l + m > C_1$, then $B(F(v_{k+l})) = f(B_{k+l}(v))$.*

More informally; the smallest ball containing $f(B_{k+l}(v))$ contains nothing besides $f(B_{k+l}(v))$.

Proof Suppose $f(z) \in \mathrm{Im}\, f(Z) = Z'$ is outside of $f(B(v_{k+l}))$. So $z \notin B(v_{k+l})$ and therefore $\rho(v, z) \geq Kr^{k-m+1}$. Consequently, for all $Z' \in B(v_{k+l})$

$$\begin{aligned}
\frac{\rho(v_{k+l}, Z')}{\rho(v_{k+l}, z)} &< r^{l+m-1}, \\
\frac{\rho'(f(v_{k+l}), f(Z'))}{\rho'(f(v_{k+l}), f(z))} &< Cr^{\frac{1}{\alpha}(l+m-1)}, \\
\frac{\operatorname{diam} f(B(v_{k+l}))}{\rho'(f(v_{k+l}), f(z))} &< KCr^{\frac{1}{\alpha}(l+m-1)}, \\
\frac{r(B(F(v_{k+l})))}{\rho'(f(v_{k+l}), f(z))} &< \widetilde{C}r^{\frac{1}{\alpha}(l+m)} \quad \text{by regularity of } v_{k+l}.
\end{aligned}$$

Since $\widetilde{C}$ is a uniform constant depending on η, K and r only, there is a C_1 such that if $l + m > C_1$, we will have

$$r(B(F(v_{k+l}))) < \frac{1}{K}\rho'(f(v_{k+l}), f(z)). \tag{1}$$

But of course

$$\begin{aligned}
\rho'(f(v_{k+l}), f(z)) &\leq K \max\{\rho'(f(v_{k+l}), F(v_{k+l})),\ \rho'(f(z), F(v_{k+l}))\} \\
&\leq K \max\{r(B(F(v_{k+l}))),\ \rho'(f(z), F(v_{k+l}))\}.
\end{aligned}$$

This and (1) imply

$$\rho'(f(z), F(v_{k+l})) > r(B(F(v_{k+l}))). \qquad \square$$

Corollary 2 *The center $\pi(F(v_{k+l}))$ of $F(v_{k+l})$ is in $f(B(v_{k+l}))$.*

Now we want to verify that the image of the upper end of a singular geodesic is the upper end of a singular geodesic with comparable length.

Lemma 5 (Upper Ends go to Upper Ends) *Suppose $v_k \in V_k$ is of type II and $v_{k+l} \in V_{k+l}$, $w \in V_{k-m}$ are the upper and the lower ends, respectively of the singular geodesic associated to v_k. There exists a uniform constant $C_2 = C_2(C, C_1, K, \eta, r)$ such that $F(v_{k+l})$ is the upper end of a singular geodesic in* Hyp(Z') *whose length L' satisfies*

$$\frac{1}{\alpha}(m+l) - C_2 \leq L' \leq \alpha(m+l) + C_2.$$

Proof Let $z \in Z \backslash B_{k+l}(v)$. Then $\rho(z, \pi(v)) \geq K r^{k-m+1}$. First of all take $C_2 \geq C_1$. Then by Corollary 2, $\exists \hat{v} \in B_{k+l}(v)$ such that $f(\hat{v}) = \pi(F(v_{k+l}))$. Now for all $z_1 \in \overline{B_{k+l}}(v)$, $z_2 \in Z \backslash B_{k+l}(v)$ we have

$$\rho(\hat{v}, z_1) < K^2 r^{k+l}, \quad \rho(\hat{v}, z_2) \geq r^{k-m+1}.$$

Thus

$$\frac{\rho(\hat{v}, z_1)}{\rho(\hat{v}, z_2)} < K^2 r^{l+m-1}, \quad \text{whence}$$
$$\frac{\rho'(f(\hat{v}), f(z_1))}{\rho'(f(\hat{v}), f(z_2))} < C K^{2/\alpha} r^{\frac{1}{\alpha}(l+m-1)},$$

which, since $r(B(F(v_{k+l}))) \asymp_r \operatorname{diam}(f(B_{k+l}(v))) \asymp_{K'} \sup_{z_1} \rho'(f(\hat{v}), f(z_1))$, gives

$$\frac{r(B(F(v_{k+l})))}{\rho'(f(\hat{v}), f(z_2))} < D r^{\frac{1}{\alpha}(l+m-1)} = \widetilde{C_2} r^{\frac{1}{\alpha}(l+m)}.$$

From this it follows that $(f(\hat{v}), p-q) \in V'_{p-q}$ for all $0 \leq q \leq \frac{1}{\alpha}(l+m) - \widetilde{C_2}$, and it is obviously singular on all these levels.

On the other hand, v_{k+l} is regular, meaning there exists a $z_3 \in B_{k+l}(v)$ with $\rho(\hat{v}, z_3) \geq r^{k+l+1}$. With $z_2 \in Z \backslash B_{k+l}(v)$ such that $\rho(\hat{v}, z_2) \leq K^2 r^{k-m}$ (exists since $w \in V_{k-m}$ is regular and $\hat{v} \in B(v_{k+l}) \subset B(w)$), we have

$$\frac{\rho(\hat{v}, z_2)}{\rho(\hat{v}, z_3)} \leq (K^2/r) \cdot r^{-(m+l)},$$

that is,

$$\frac{\rho'(f(\hat{v}), f(z_2))}{\rho'(f(\hat{v}), f(z_3))} \leq C(K^2/r)^{\alpha} \cdot r^{-\alpha(m+l)},$$

which bounds the length of the singular geodesic descending from $F(v_{k+l})$ by $\alpha(m+l) + \widehat{C_2}$. Setting $C_2 := \max\{C_1, \widetilde{C_2}, \widehat{C_2}\}$ proves the lemma. □

So if $w v_{k-m+1} \cdots v_k \cdots v_{k+l}$ is a singular geodesic in Hyp(Z), then $F(v_{k+l})$ is the upper end of a singular geodesic in Hyp(Z') with controlled length. Now we want to know how $F(w)$ and the lower end of the image singular geodesic are related, cf. Fig. 3.

Lemma 6 (Lower Ends go roughly to Lower Ends) *Suppose $v_k \in V_k$ is singular. If v_k is of type II, let $w v_{k-m+1} \cdots v_k \cdots v_{k+l}$ be the singular segment in* Hyp(Z) *determined by v_k with lower end $w \in V_{k-m}$ and upper end $v_{k+l} \in V_{k+l}$, and let $w' v'_{p-q+1} \cdots F(v_{k+l})$ be the*

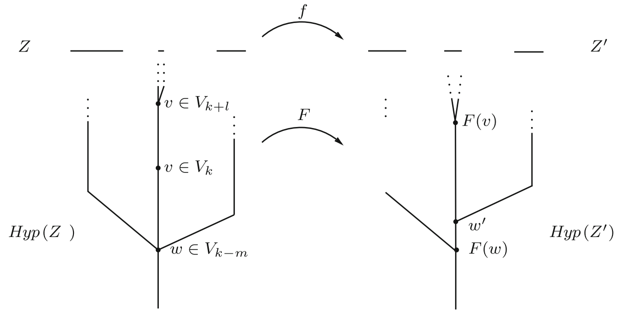

Fig. 3 The distance between w' and *F(w)* is uniformly bounded

singular segment in $\mathrm{Hyp}(Z')$ *associated to* $F(v_{k+l}) \in V'_p$ *according to* Lemma 5. *If* v_k *is of type* III *and* $wv_{k-n}\cdots v_k\cdots$ *the associated singular ray in* $\mathrm{Hyp}(Z)$, *denote by* w' *the root of the singular ray in* $\mathrm{Hyp}(Z')$ *associated to* $f(\pi(v_k))$.

There is a uniform constant $C_3 = C_3(\eta, K, r)$ *such that* $|w'F(w)| \leq C_3$.

Proof We show it first for v of type II. We may assume that $l+m > C_1$, for if not, Lemma 5 says that w' is uniformly close to $F(v_{k+l})$, and the fact that $\operatorname{diam} f(B(v_{k+l}))$ is uniformly comparable to $\operatorname{diam} f(B(w))$ (and the sets intersect) shows that $F(w)$ uniformly close to $F(v_{k+l})$.

Now $F(w)$ is by definition the smallest ball containing $f(B_{k-m}(w))$. In particular $f(B_{k+l}(v)) \subset B(F(w))$, so that $B(F(w)) \cap B(w') \neq \emptyset$. Now the distance between vertices whose associated balls intersect is roughly equal to their level distance (cf. [3], Lemma 6.2.7). Hence we must show that $l(w') \doteq l(F(w))$, which is the case iff $r(B(w')) \asymp r(F(w))$, iff

$$\operatorname{diam} f(B_{k-m}(w)) \asymp r(B(w')). \tag{2}$$

Now,

$$r(B(w')) \asymp_r \inf_{Z' \in Z' \setminus f(B_{k+l}(v))} \rho'(Z', \pi(F(v_{k+l}))).$$

But we know (Corollary 2) that the center of the ball $F(v_{k+l})$ is given by $f(\hat{v})$ for some $\hat{v} \in B_{k+l}(v)$. Since f is bijective we can write

$$r(B(w')) \asymp_r \inf_{z \in Z \setminus B_{k+l}(v)} \rho'(f(z), f(\hat{v})). \tag{3}$$

For the l.h.s. of (2) we have

$$\operatorname{diam} f(B_{k-m}(w)) \asymp_K \sup_{z \in B_{k-m}(w)} \rho'(f(\hat{v}), f(z)) \tag{4}$$

because $\hat{v} \in B_{k+l}(v) \subset B(w)$.

With (3) and (4), (2) becomes

$$\sup_{z \in B_{k-m}(w)} \rho'(f(\hat{v}), f(z)) \asymp \inf_{z \in Z \setminus B_{k+l}(v)} \rho'(f(\hat{v}), f(z)) \tag{5}$$

Simplifying further, for any $z \in B(w) \setminus B(v_{k+l})$ we have $\frac{\rho(v,\hat{v})}{\rho(v,z)} < 1$, whence by quasi-symmetry

$$\sup_{z \in B(w)} \rho'(f(\hat{v}), f(z)) \leq D \cdot \sup_{z \in B(w) \setminus B(v_{k+l})} \rho'(f(\hat{v}), f(z)),$$

for a uniform D. This gives

$$\sup_{z\in B(w)} \rho'(f(\hat{v}), f(z)) \asymp \sup_{z\in B(w)\backslash B(v_{k+l})} \rho'(f(\hat{v}), f(z)). \tag{6}$$

Likewise we get

$$\inf_{z\in Z\backslash B(v_{k+l})} \rho'(f(\hat{v}), f(z)) \asymp \inf_{z\in B(w)\backslash B(v_{k+l})} \rho'(f(\hat{v}), f(z)), \tag{7}$$

for pick a $\hat{z} \in B(w)\backslash B(v_{k+l})$ such that for some uniform E

$$\frac{\rho(\hat{v}, \hat{z})}{\rho(\hat{v}, z)} \leq E \quad \forall z \in Z\backslash B(v_{k+l}).$$

Then

$$\eta(E) \cdot \inf_{z\in Z\backslash B(v_{k+l})} \rho'(f(\hat{v}), f(z)) \geq \rho'(f(\hat{v}), f(\hat{z})) \geq \inf_{z\in B(w)\backslash B(v_{k+l})} \rho'(f(\hat{v}), f(z)),$$

from which (7) follows immediately.

With (6) and (7), (5) follows if we prove

$$\inf_{z\in B(w)\backslash B(v_{k+l})} \rho'(f(\hat{v}), f(z)) \asymp \sup_{z\in B(w)\backslash B(v_{k+l})} \rho'(f(\hat{v}), f(z)).$$

One direction is trivial and we just have to show

$$\sup_{z\in B(w)\backslash B(v_{k+l})} \rho'(f(\hat{v}), f(z)) \leq H \cdot \inf_{z\in B(w)\backslash B(v_{k+l})} \rho'(f(\hat{v}), f(z)), \tag{8}$$

for some uniform constant H. But in fact, for any $z \in B(w)\backslash B(v_{k+l})$ we have $r^{k-m+1} \leq \rho(\hat{v}, z) \leq K^2 r^{k-m}$, thus there is a uniform $\widetilde{H}$ such that

$$\frac{\rho(\hat{v}, z_1)}{\rho(\hat{v}, z_2)} \leq \widetilde{H} \; \forall z_1, z_2 \in B(w)\backslash B(v_{k+l}), \text{ and hence}$$

$$\frac{\rho'(f(\hat{v}), f(z_1))}{\rho'(f(\hat{v}), f(z_2))} \leq \eta(\widetilde{H}).$$

This implies (8) and thereby the lemma for v_k of type II.

The argument for v_k of type III is analoguous. $B(w')$ and $B(F(w))$ again intersect, so we must estimate their level difference. Denote $\hat{z} := \pi(v_k)$. $F(w)$ is the smallest ball containing $f(B(w))$, while the radius of $B(w')$ is determined by when a ball around $f(\hat{z})$ starts to contain points in $Z'\backslash\{f(\hat{z})\}$.

In formulas

$$r(B(F(w))) \asymp_{C(K,r)} \operatorname{diam} f(\overline{B}(w)) \quad r(w') \asymp_{D(K,r)} \inf_{z\in Z\backslash\{\hat{z}\}} \rho'(f(z), f(\hat{z})),$$

where $C(K, r)$ and $D(K, r)$ are appropriate expressions involving only K and r. Since $\operatorname{diam} f(B(w)) \asymp_{E(r,K')} \sup_{z\in B(w)} \rho'(f(\hat{z}), f(z))$, the claim follows once we show

$$\sup_{z\in B(w)} \rho'(f(\hat{z}), f(z)) \asymp_{\widetilde{C_4}(K,r)} \inf_{z\in Z\backslash\{\hat{z}\}} \rho'(f(\hat{z}), f(z)). \tag{9}$$

Now the same steps as in the proof of (5) yield the lemma for v_k of type III. □

So far we have only defined where F maps regular vertices. We are now in a position to extend the domain of F to all of $\operatorname{Hyp}(Z)$.

$v \in V_k$ is of type I — $F(v) \in \mathrm{Hyp}(Z')$ is defined to be a vertex of highest level w' such that $f(B(v)) \subset B(w')'$.

$v \in V_k$ is of type II — $v = v_k \in V_k$ lies on a singular geodesic $wv_{k-m+1} \cdots v_{k+l}$ with lower and upper ends $w \in V_{k-m}$, $v = v_{k+l} \in V_{k+l}$. In case $l + m < \alpha C_2$, set $F(v) := F(w)$. If $l + m \geq \alpha C_2$, then $F(v_{k+l}) \in V'_p$ is the upper end of a singular geodesic whose length L' satisfies $\frac{1}{\alpha}(l+m) - C_2 \leq L' \leq \alpha(l+m) + C_2$ (Lemma 5) and if $w' \in V'_{p-L}$ denotes the lower end of this singular geodesic, then $|F(w)w'| \leq C_3$ (Lemma 6). Let $L = l + m$. In this case define $F(v \in V_k)$ to be a vertex v' on the singular geodesic from w' to $F(v_{k+l})$ for which $|w'v'| \doteq_1 \frac{L'}{L}|wv|$.

$v \in V_k$ is of type III — $v \in V_k$ lies on a singular ray in $\mathrm{Hyp}(Z)$ going to $\pi(v) \in Z$. Since $|Z| \geq 2$, this singular ray has a regular lower end $w \in V_{k-m}$. Since f is a homeomorphism, $f(v)$ is isolated in Z', thus there is a singular ray in $\mathrm{Hyp}(Z')$ starting at some regular $w' \in V'_p$. $F(v)$ is defined as the (unique) vertex $v' \in V'_{p+m}$ on this ray. Equivalently, $F(v)$ is the vertex v' on the singular ray in $\mathrm{Hyp}(Z')$ same distance from w' as v has from w.

This defines F on the whole vertex set $\mathcal{V}$, and up to a rough isometry, F is then well-defined on all of $\mathrm{Hyp}(Z)$.

Theorem 6 *The map* $F : \mathrm{Hyp}(Z) \to \mathrm{Hyp}(Z')$ *described above is a quasi-isometry, and* $\partial_\infty F = f$.

Proof We first show that F is Lipschitz. Since $\mathrm{Hyp}(Z)$ is geodesic, this follows if we show that the distance $|F(v)F(w)|$ is uniformly bounded for neighboring $v, w \in \mathrm{Hyp}(Z)$. Now if v, w are both of type I, it follows by standard arguments (such as those used in the proof of Theorem 7.2.1 in [3]) that the level difference of $F(v)$ and $F(w)$ is uniformly bounded. If, w.l.o.g. v is of type I and w of type II, Lemmas 5 and 6 (or the definition of F if w is not on a long enough singular geodesic) imply that $|v'w'|$ uniformly bounded. If v of type I and w of type III, Lemma 6 does the job. A vertex of type II never neighbors a vertex of type III. This proves that F is Lipschitz.

Next define a map $G : \mathrm{Hyp}(Z') \to \mathrm{Hyp}(Z)$ corresponding to $f^{-1} : Z' \to Z$ in the same way F was defined (and with the same choice of vertex systems $\mathcal{V}, \mathcal{V}'$). Of course G is then also Lipschitz. We show $G \circ F \doteq \mathrm{id}_{\mathrm{Hyp}(Z)}$.

v of type I — By definition $B(G \circ F(v)) \supset B(v)$. In particular, the balls intersect. Their distance is uniformly bounded iff the diameters of these sets are uniformly comparable. But this follows from the facts that $f(B(v)) \subset B(F(v))$, diam $f(B(v))$ is uniformly comparable to diam $B(F(v))$, and that f^{-1} is quasisymmetric. The doubtful reader is referred to [9], Theorem 2.5, which describes exactly this situation.

v of type II — We have a singular geodesic $wv_{k-m+1} \cdots v = v_k \cdots v_{k+l}$ with lower end $w \in V_{k-m}$ and upper end v_{k+l}. By Lemma 5 applied twice to F and then G, there is a uniform constant C_5 such that if $l + m > C_5$, not only $F(v_{k+l})$ is an upper end of a singular geodesic in $\mathrm{Hyp}(Z)$ but even $G(F(v_{k+l}))$ is still the upper end of singular geodesic in $\mathrm{Hyp}(Z)$. $F(v_{k+l})$, as usual, is a smallest ball containing $f(B_{k+l}(v))$. But by Lemma 4 $B(F(v_{k+l})) = f(B_{k+l}(v))$. In particular, $G(F(v_{k+l}))$, being the smallest ball containing $f^{-1}(B(F(v_{k+l})))$, is just $B_{k+l}(v)$. In other words,

$G(F(v_{k+l})) = v_{k+l}$. By definition of F and G it is now obvious that $G(F(v))$ is uniformly close to v.

If the singular geodesic $wv_{k-m+1}\cdots v = v_k \cdots v_{k+l}$ is shorter than C_5, then v is in particular uniformly close to a type I vertex, namely w (or v_{k+l}). The Lipschitz property of F and G and the fact that $G(F(w))$ is uniformly close to w imply that $G(F(v))$ is uniformly close to v.

v of type III $\pi(v) = z$, an isolated point in Z. $f(z)$ is an isolated point in Z' and by definition of F, the ray in Hyp(Z) associated to z, on which v lies, is mapped one-to-one onto the ray in Hyp(Z') associated to $f(z)$. But then G maps this ray back in one-to-one fashion to the ray associated to $f^{-1}(f(z)) = z$. So in this case we have in fact $v = G(F(v))$.

This proves $G \circ F \doteq \mathrm{id}_{\mathrm{Hyp}(Z)}$. Since the domain of G is all of Hyp(Z'), it follows that $F(\mathrm{Hyp}(Z))$ is cobounded in Hyp(Z'), thus F is a quasi-isometry.

It remains to show that $\partial_\infty F = f$. By [3], Theorem 5.2.17, we know that F *does* induce a homeomorphism $\partial_\infty F : Z \to Z'$. So take a sequence $\{v_i\}$ of vertices converging to $z \in Z$. We have $\pi(v_i) \to z$ in (Z, ρ). Since the limit of the sequence $\{F(v_i)\}$ does not depend on the representative $\{v_i\} \in z$, we may take the latter such that $B(v_{i+1}) \subset B(v_i)$ (cf. [3], Lemma 6.3.2) Then $\{F(v_i)\}$ converges to some $Z' \in Z'$. In particular, $l'(F(v_i)) \overset{i\to\infty}{\longrightarrow} \infty$. Since $\rho'(f(\pi(v_i)), \pi(F(v_i))) \le Kr^{l'(F(v_i))}$ and $f(\pi(v_i)) \to f(z)$, we get $\pi(F(v_i)) \to f(z)$ in Z' and this implies $\partial_\infty F(z) = f(z)$. $\square$

6 Extension for inversions

There is a good reason why one would not be satisfied with describing the quasisymmetric structure of the boundary, but would rather have a result on its quasimoebius structure. Namely, there is in general no uniform constant L such that id : $\partial_\infty^{a,o} X \to \partial_\infty^{a,o'} X$ is L-bilipschitz for any $o, o' \in X$. However, there *is* a uniform L (depending on a, δ) such that it is L-bilipschitz-quasi*moebius*. In other words, the ratio of a triple of boundary points is not a uniform quantity, whereas the cross-ratio of a quadruple is. For more on this we refer to [7], Theorem 8.1. This motivates us to look for an extension theorem for quasimoebius maps in the spirit of the Poincaré extension theorems for classical hyperbolic space.

In this section we prove that the hyperbolic approximation of a bounded quasimetric space (Z, ρ) is roughly isometric to the hyperbolic approximation (with the same parameters) of the extended quasimetric space (Z, ρ') where ρ' is the inversion at a point in Z of ρ. This result will be combined with theorem 5 to give the desired Moebius extension.

Theorem 7 *Let (Z, ρ) be a bounded complete quasi-metric space and ρ' the quasi-metric obtained from ρ by inversion in a point $\omega \in Z$,*

$$\rho'(a, b) := \frac{\rho(a, b)}{\rho(a, \omega)\rho(b, \omega)}.$$

Then the (truncated) hyperbolic approximation of (Z, ρ) is roughly isometric to the hyperbolic approximation of (Z, ρ'). More precisely, for every $r \in (0, 1)$ there exists a rough isometry $F : \mathrm{Hyp}_r(Z, \rho) \to \mathrm{Hyp}_r(Z, \rho')$ that induces the identity in $\partial_\infty \mathrm{Hyp}(Z) = Z$.

This theorem is trivial for $Z = \{z, \omega\}$, so we shall assume $|Z| \ge 3$.

Remark 4 The proof of this theorem basically consists of a series of uniform comparability statements, $\cdot \asymp \cdot$, all of which remain true if the boundary quasimetrics are replaced by ones that are bilipschitz equivalent to them. In particular, the theorem allows us to conclude, via the bilipschitz extension Theorem 4, that $\mathrm{Hyp}_{1/a}(\partial_\infty^{a,b_1(\omega)} X)$ is roughly isometric to $\mathrm{Hyp}_{1/a}(\partial_\infty^{a,b_2(\omega)} X)$, where $b_1(\omega)$, $b_2(\omega)$ are two arbitrary Busemann functions at ω. This fact will be needed in the proof of $(III) \Rightarrow (I)$ in Theorem 10.

Note that if (Z, ρ) is K-quasimetric, then (Z, ρ') is K^2-quasimetric. Throughout this section we assume that both approximations $\mathrm{Hyp}(Z, \rho)$, $\mathrm{Hyp}(Z, \rho')$ are done with respect to the same K. Since the rough isometry class of the approximations does not depend on the K used, this poses no danger. Moreover, we may assume $r < 1/K^3$, since for all other values of r, Hyp_r is obtained by scaling the graphs $\mathrm{Hyp}_{r'}(Z, \rho)$, $\mathrm{Hyp}_{r'}(Z, \rho')$, where $r' < 1/K^3$, by the same factor.

In addition, it turns out to be advantageous to work with a special choice of vertex system $\mathcal{V}$ for $\mathrm{Hyp}(Z, \rho)$. Namely we require that $\mathcal{V}$ be hereditary and the root o be centered at the inversion point ω, $\pi(o) = \omega$. In particular, we then have a canonical "ray to ω" in $\mathrm{Hyp}(Z, \rho)$, namely the radial geodesic ray consisting of all vertices centered at ω. We will often refer to this ray as *the ray* $o\omega$.

The idea of the definition for F is to do the same as for quasi-symmetric maps whenever ω is not involved, and "invert the orientation" on the ray $o\omega$. This corresponds to the fact that the inversion restricted to $Z \backslash O$, where O is any neighborhood of ω, is a PQ-symmetry onto its image because it is a Moebius map between bounded spaces (cf. Lemma 12).

We define the map F.

Definition 13 *I* If v is regular with $\pi(v) = \omega$ and $v \neq o$, set $F(v) :=$ any vertex w of highest level in $\mathrm{Hyp}(Z, \rho')$ such that $B^{\rho'}_{Kr^{l(w)}}(w)$ contains $B^{\rho}_{Kr^{l(v)+1}}(\pi(v))^c$.

II If v is a horizontal neighbor to a vertex $\tilde{v}$ as in I, set $F(v) := F(\tilde{v})$.

III If $v \neq o$ is regular and neither as in I nor II, set $F(v) :=$ any vertex w of highest level in $\mathrm{Hyp}(Z, \rho')$ such that $B^{\rho'}(w) \supset B^{\rho}_{Kr^{l(v)}}(\pi(v))$.

IV For the root o, if the immediate radial successor v to o on the ray $o\omega$ is regular, set $F(o) := F(v)$. If this v is not regular, then $Z \backslash B(v)$ is separated from the rest of Z (in the sense that the two sets have positive distance) and the same is the case in (Z, ρ'). Furthermore, there is a branch point (cf. [3], p. 72) in $\mathrm{Hyp}(Z, \rho')$ for $\{B := B^{\rho}_{Kr^{l(o)+1}}(\omega), Z \backslash B\}$. In this case set $F(o) =$ such a branch point.

V If v is singular and lies on a singular segment $w_1 w_2$ in $\mathrm{Hyp}(Z, \rho)$, map it to an appropriate vertex on the singular segment associated to $w_1 w_2$ in $\mathrm{Hyp}(Z, \rho')$, cf. Lemma 9.

VI If v is singular and lies on a singular ray wz in $\mathrm{Hyp}(Z, \rho)$, map v to an appropriate vertex on the singular ray in $\mathrm{Hyp}(Z, \rho')$ associated to the ray wz, cf. Lemma 10.

The verification that F is a rough isometry is straightforward but a bit tedious. We first show $|F(v)F(w)| \doteq |vw|$ for v, w from a cobounded subset of the set of regular vertices, Lemma 8. Then we can extend it to all v, w regular. Afterwards we show well-behavedness of singular segments and rays, Lemmata 9 and 10, respectively.

Lemma 7 *Let v, w be any regular vertices in* $\mathrm{Hyp}(Z, \rho)$. *Then*

$$|vw| \doteq \log_r \left(\frac{\mathrm{diam}(B(v))\mathrm{diam}(B(w))}{\sup \rho(z_v, z_w)^2} \right).$$

Proof There is a geodesic connecting v to w that has either exactly one or exactly two points of lowest level (cf. [3], Lemma 6.2.6). In either case, there is a branch point u for $\{v, w\}$ with distance at most one from any lowest level vertex. Then

$$|vw| \doteq_1 (l(v) - l(u)) + (l(w) - l(u)).$$

But $l(v) \doteq \log_r(\operatorname{diam} B(v))$ by regularity of v (the error constant depending on K, r), and the same for w.

Now $B(v) \cup B(w) \subset B(u)$ by definition. On the other hand, any vertex t such that $B(v) \cup B(w) \subset B(t)$ is uniformly close (error 1) to a cone point by [3] Lemma 6.2.1. Take t to be any vertex of highest level satisfying $B(v) \cup B(w) \subset B(t)$, then t is uniformly close to a (and hence, any) branch point. But then $\operatorname{diam}(B(t)) \asymp \sup \rho(z_v, z_w)$. The lemma follows. □

Lemma 8 *Let v, w be regular vertices in* $\operatorname{Hyp}(Z, \rho)$ *which if centered at ω or horizontally connected to $o\omega$ are at least two levels above the root. Then $|F(v)F(w)| \doteq |vw|$.*

Proof For the proof we show that

$$\frac{\operatorname{diam}_{\rho'}(B^{\rho'}(F(v)))\operatorname{diam}_{\rho'}(B^{\rho'}(F(w)))}{\sup \rho'(z_{F(v)}, z_{F(w)})^2} \asymp \frac{\operatorname{diam}_\rho(B^\rho(v))\operatorname{diam}_\rho(B^\rho(w))}{\sup \rho(z_v, z_w)^2}, \tag{10}$$

which implies the claim by Lemma 7. Here the notation z_v, z_w is supposed to suggest that the sup is taken over all $z_v \in B(v)$, $z_w \in B(w)$ and likewise for $F(v), F(w)$.

If $\pi(v) = \pi(w) = \omega$, we have $\operatorname{diam}_{\rho'}(B(F(v))) \asymp \frac{1}{\operatorname{diam}_\rho(B(v))}$. (10) simplifies to

$$\sup \rho'(z_{F(v)}, z_F(w)) \asymp \frac{\sup \rho(z_v, z_w)}{\operatorname{diam}(B(v))\operatorname{diam}(B(w))},$$

both sides of which compare uniformly to $1/\operatorname{diam}(B(v))$ (if w.l.o.g. $l(v) \geq l(w)$). By II of Definition 13, the same argument gives (10) for vertices horizontally connected to the ray $o\omega$. So suppose $\pi(v) = \omega$ and w not horizontally connected to the ray. Then

$$\rho'(z, w) = \frac{\rho(z, w)}{\rho(z, \omega)\rho(w, \omega)} \asymp \frac{\rho(z, w)}{\rho(w, \omega)^2} \quad \forall z \in B(w),$$

whence

$$\operatorname{diam}_{\rho'}(B(w)) \asymp \frac{\operatorname{diam}_\rho(B(w))}{\rho(w, \omega)^2}.$$

(10) then becomes

$$\frac{\operatorname{diam}_{\rho'}(B(F(v)))}{\sup \rho'(z_{v_{+1}^c}, z_w)^2 \rho(w, \omega)^2} \asymp \frac{\operatorname{diam}_\rho(B(v))}{\sup \rho(z_v, z_w)^2}, \tag{11}$$

where $z_{v_{+1}^c}$ suggests elements in $B(v_{+1})^c := B^\rho_{Kr^{l(v)+1}}(v)^c$. Since $\operatorname{diam}_{\rho'}(B(F(v))) \asymp 1/\operatorname{diam}_\rho(B(v))$, (11) is equivalent to

$$\sup \rho'(z_{v_{+1}^c}, z_w)\rho(w, \omega) \asymp \frac{\sup \rho(z_v, z_w)}{\sup \rho(z_v, \omega)}.$$

Thus we must show

$$\sup \frac{\rho(z_{v_{+1}^c}, z_w)}{\rho(z_{v_{+1}^c}, \omega)\rho(z_w, \omega)} \cdot \rho(w, \omega) \asymp \frac{\sup \rho(z_v, z_w)}{\sup \rho(z_v, \omega)}$$

which, since $\rho(z_w, \omega) \asymp \rho(w, \omega)$, finally becomes

$$\sup \frac{\rho(z_{v_{+1}^c}, z_w)}{\rho(z_{v_{+1}^c}, \omega)} \asymp \frac{\sup \rho(z_v, z_w)}{\sup \rho(z_v, \omega)}. \tag{12}$$

We prove (12), which implies the lemma in case v on the ray, w not horizontally connected to the ray. We show first that the l.h.s. of (12) is $\geq \frac{1}{K}$. Since v is regular $\exists z_1 \in B(v_{+1})^c$ with $\rho(\omega, z_1) < Kr^k$, and since v is at least 2 from the root, there also exists $z_2 \in B(v_{+1})^c$ with $\rho(\omega, z_2) \geq r^{k-1}$. Now suppose for all z_1 with $\rho(\omega, z_1) < Kr^k$, where $k = \ell(v)$.

$$\rho(z_1, z_w) < \frac{1}{K}\rho(z_1, \omega) \; \forall z_w \in B(w).$$

Then $\rho(z_1, \omega) \asymp_K \rho(z_w, \omega)$. But now z_2 is much farther from ω than z_1, hence

$$\rho(z_2, \omega) \asymp_K \rho(z_2, z_w) \;\; \forall z_w \in B(w).$$

This shows that the l.h.s. of (12) is $\geq 1/K$ in any case.

Next suppose

$$\sup \frac{\rho(z_{v_{+1}^c}, z_w)}{\rho(z_{v_{+1}^c}, \omega)} > K^4. \tag{13}$$

Then since necessarily $\rho(z_{v_{+1}^c}, z_w) \asymp_K \rho(z_w, \omega)$ for $z_{v_{+1}^c}, z_w$ such that the sup is (almost) attained,

$$\rho(z_w, \omega) > K^3 \rho(z_{v_{+1}^c}, \omega). \tag{14}$$

That is, when $z_w, z_{v_{+1}^c}$ are taken so that the sup is (almost) attained, z_w will be much farther away from ω than $z_{v_{+1}^c}$. We want to know that then $z_{v_{+1}^c}$ may as well be taken in $B(v)$, thus we must show that if $z_2 \in B(v)$ is arbitrary, then the quantity

$$\frac{\rho(z_2, z_w)}{\rho(z_2, \omega)},$$

where the z_w is the same as above, is not smaller (or at least not by much) than when z_2 is replaced by $z_{v_{+1}^c}$. So pick $z_2 \in B(v)$ arbitrary. We may suppose $\rho(z_2, \omega) < \rho(z_{v_{+1}^c}, \omega)$, otherwise $z_{v_{+1}^c}$ would already be in $B(v)$ and we are done. So then

$$\rho(z_{v_{+1}^c}, z_2) \leq K\rho(z_{v_{+1}^c}, \omega) \overset{(14)}{<} \frac{1}{K^2}\rho(z_w, \omega) \leq \frac{1}{K}\rho(z_{v_{+1}^c}, z_w),$$

whence

$$\rho(z_{v_{+1}^c}, z_w) \asymp_K \rho(z_2, z_w).$$

Since $\rho(z_{v_{+1}^c}, \omega) > \rho(z_2, \omega)$ it thus follows that

$$\frac{\rho(z_2, z_w)}{\rho(z_2, \omega)} > \frac{1}{K} \frac{\rho(z_{v_{+1}^c}, z_w)}{\rho(z_{v_{+1}^c}, \omega)}.$$

It follows that the claimed uniform comparability of (12) holds.

It remains to prove (12) when

$$\frac{1}{K} \leq \sup \frac{\rho(z_{v_{+1}^c}, z_w)}{\rho(z_{v_{+1}^c}, \omega)} \leq K^4.$$

In fact we show more, namely

$$\sup \frac{\rho(z_{v^c_{+1}}, z_w)}{\rho(z_{v^c_{+1}}, \omega)} \asymp_{K^4} 1 \implies \frac{\sup \rho(z_v, z_w)}{\sup \rho(z_v, \omega)} \asymp 1. \tag{15}$$

The assumption on the l.h.s. means in particular that for any choice of $z_{v^c_{+1}}$, every z_w lies rather close to $z_{v^c_{+1}}$. Quantitatively speaking we have

$$\rho(z_w, \omega) \leq K^5 \min\{\rho(z_{v^c_{+1}}, z_w), \rho(z_{v^c_{+1}}, \omega)\} \quad \forall z_w, z_{v^c_{+1}}. \tag{16}$$

Now since v is regular, there is a $z_{v^c_{+1}}$ with $\rho(z_{v^c_{+1}}, \omega) \leq K r^{l(v)}$. Then by (16), $\rho(z_w, \omega) \leq K^6 r^{l(v)}$.

On the other hand $B(w)$ must not contain ω, so $\rho(z_w, \omega) \geq K r^{l(w)}$. It thus follows that $l(w) \geq l(v)$ up to a uniform error, or in words that $B(w)$ is smaller than $B(v)$ up to a uniform factor.

But then

$$\sup \rho(z_v, z_w) \asymp \sup \rho(z_v, \omega).$$

In addition,

$$\sup \rho(z_v, \omega) \asymp \sup \rho(z_v, w),$$

since $B(w)$ is contained within the ball of radius $K^6 r^{l(v)}$ around ω. This proves (15).

It remains to prove the lemma for v, w both not horizontally connected to nor on the ray.

We start again with (10),

$$\frac{\operatorname{diam}_{\rho'}(B(F(v)))\operatorname{diam}_{\rho'}(B(F(w)))}{\sup \rho'(z_{F(v)}, z_{F(w)})^2} \asymp \frac{\operatorname{diam}_{\rho}(B(v))\operatorname{diam}_{\rho}(B(w))}{\sup \rho(z_v, z_w)^2}.$$

Since v is not connected to the ray, we get, just as in the case above

$$\rho'(z, v) = \frac{\rho(z, v)}{\rho(z, \omega)\rho(v, \omega)} \asymp_K \frac{\rho(z, v)}{\rho(v, \omega)^2} \quad \forall z \in B(v)$$

and thus

$$\operatorname{diam}_{\rho'}(B(v)) \asymp \frac{\operatorname{diam}_{\rho}(B(v))}{\rho(v, \omega)^2}.$$

The same estimate also holds for $\operatorname{diam}_{\rho'}(B(w))$. (10) becomes

$$\frac{\operatorname{diam}_{\rho}(B(v))\operatorname{diam}_{\rho}(B(w))}{\rho(v, \omega)^2 \rho(w, \omega)^2 \sup \rho'(z_v, z_w)^2} \asymp \frac{\operatorname{diam}_{\rho}(B(v))\operatorname{diam}_{\rho}(B(w))}{\sup \rho(z_v, z_w)^2},$$

which is equivalent to

$$\sup \rho'(z_v, z_w)\rho(v, \omega)\rho(w, \omega) \asymp \sup \rho(z_v, z_w).$$

This follows if we can show that

$$\rho'(z_v, z_w)\rho(v, \omega)\rho(w, \omega) \asymp_C \rho(z_v, z_w) \quad \forall z_v, z_w \tag{17}$$

for some uniform constant C. But (17) is equivalent to

$$\frac{\rho(z_v, z_w)}{\rho(z_v, \omega)\rho(z_w, \omega)}\rho(v, \omega)\rho(w, \omega) \asymp_C \rho(z_v, z_w).$$

It thus suffices to show

$$\rho(z_v, \omega) \asymp \rho(v, \omega)$$
$$\rho(z_w, \omega) \asymp \rho(w, \omega),$$

and these estimates hold because $\rho(z_v, \omega) > Kr^{l(v)}$, so in

$$\{\rho(z_v, \omega), \rho(v, \omega), \rho(v, z_v)\}$$

the minimum will always, that is, for all $z_v \in B(v)$, be $\rho(v, z_v)$, thus $\rho(z_v, \omega) \asymp_K \rho(v, \omega)$. The same holds for w. The lemma follows. □

Corollary 3 *Let v, w arbitrary regular vertices. Then $|F(v)F(w)| \doteq |vw|$.*

Proof This follows from Lemma 8. □

Lemma 9 *Let v, w be the top and lower ends, respectively of a singular segment in* $\mathrm{Hyp}(Z, \rho)$. *Then $F(v), F(w)$ are uniformly close to the ends of a singular segment in* $\mathrm{Hyp}(Z, \rho')$ *of roughly the same length.*

Proof First assume that v is not on the ray $o\omega$ and not horizontally connected to it. Consider $z_0, z_1 \in B(v)$ and $z_2 \in B(v)^c$. Then

$$\frac{\rho'(z_0, z_1)}{\rho'(z_0, z_2)} = \frac{\rho(z_0, z_1)\rho(z_2, \omega)}{\rho(z_1, \omega)\rho(z_0, z_2)}.$$

This cannot be (much) larger than $\rho(z_0, z_1)/\rho(z_0, z_2)$, which implies that $F(v)$ is the top end of a singular segment of length $\dot{\geq}|vw|$.

If we can prove that $\ell(F(w)) < \ell(F(v))$, then a geodesic joining $F(w)$ to $F(v)$ will reach $F(v)$ from below, thus has to go through the singular segment. Since $|F(v)F(w)| \doteq |vw|$ by Lemma 8, the lemma follows.

Now if w is neither on the ray $o\omega$ nor horizontally connected to it, then

$$\frac{\rho'(v, z_v)}{\rho'(w, z_w)} = \frac{\rho(v, z_v)}{\rho(w, z_w)} \cdot \frac{\rho(w, \omega)\rho(z_w, \omega)}{\rho(z_v, \omega)\rho(v, \omega)} \asymp_{K^2} \frac{\rho(v, z_v)}{\rho(w, z_w)},$$

whence $l(w) \dot{\leq} l(v)$. Similar estimates hold in case w is connected to or on the ray $o\omega$, that is, the ρ'-diameter of $B(w_{+1})^c$ is much larger than that of $B(v)$, where , again, the notation $B(w_{+1})$ means the ball associated to $(\pi(w), l(w) + 1)$, i.e. $B_{Kr^{l(w)+1}}(\pi(w))$. This proves the lemma in case v is not horizontally connected to, nor on the ray.

Finally, if $\pi(v) = \omega$, then also $\pi(w) = \omega$ or $F(w) = F(\tilde{w})$ with $\pi(\tilde{w}) = \omega(\tilde{w}$ being a horizontal neighbor to w on the ray). It follows immediately by definition of ρ' that there is a singular segment of roughly the same length between $F(v)$ and $F(w)$ (as long as $w \neq o$, but in this case simply apply the definition of F). □

Now we show that a root of a singular ray in $\mathrm{Hyp}(Z, \rho)$ is mapped uniformly close to the root of a singular ray in $\mathrm{Hyp}(Z, \rho')$.

Lemma 10 *There is a one-to-one correspondence between singular rays in* $\mathrm{Hyp}(Z, \rho)$ *and* $\mathrm{Hyp}(Z, \rho')$ *and a root of a singular ray in* $\mathrm{Hyp}(Z, \rho)$ *is mapped uniformly close to a (hence, any) root of the associated singular ray in* $\mathrm{Hyp}(Z, \rho')$, *with the exception of a singular ray in* $\mathrm{Hyp}(Z, \rho)$ *going to ω, which is mapped to a singular ray "downwards" to ∞ in* $\mathrm{Hyp}(Z, \rho')$.

Proof That there is a one-to-one correspondence is clear because every singular ray corresponds to an isolated point in the boundary, and $\mathrm{id}|_{Z\backslash\{\omega\}}$ is a homeomorphism onto its image, so maps isolated points to isolated points, and if there is a singular ray to ω then $(Z\backslash\{\omega\}, \rho')$ is bounded, so there will be an associated singular ray descending to ∞ in $\mathrm{Hyp}(Z, \rho')$. We just need to argue that the root of a ray associated to z in $\mathrm{Hyp}(Z, \rho)$ is mapped close to the root of the ray associated to z in $\mathrm{Hyp}(Z, \rho')$. Assume first that if v is a root of the ray associated to z, then either v is not connected to nor on the ray $o\omega$, or if it is on the ray, then it is at least two levels above o.

Now note $B(F(v))$ contains z by definition. It therefore suffices to show that the level of $F(v)$ is roughly the same as that of the root q of the ray associated to z in $\mathrm{Hyp}(Z, \rho')$. Now if v is not connected to nor on the ray $o\omega$, then $\mathrm{diam}_{\rho'}(B(v)) \asymp \mathrm{diam}_{\rho}(B(v))/\rho(v, \omega)^2$ and similarly $\inf_{Z'\neq z} \rho'(z, z') \asymp \inf \rho(z, z')/\rho(v, \omega)^2$, hence the levels of q and $F(v)$ agree up to uniform error. If on the other hand v is centered at ω

$$\inf_{z'} \rho'(z, z') = \inf \frac{\rho(z, z')}{\rho(z, \omega)\rho(z', \omega)} \geq \frac{1}{\min\{\rho(z, \omega), \rho(z', \omega)\}},$$

and since v is at least two levels above the root, there exists z' such that $\rho(z', \omega) > K\rho(z, \omega)$, i.e. $\rho(z, \omega) \asymp_K \rho(z, z')$. It follows that $\inf_{z'} \rho'(z, z') \asymp 1/\rho(z, \omega)$. The same argument yields that $\mathrm{diam}_{\rho'}(B(v_{+1}))^c \asymp 1/\rho(z, \omega)$.

For the exceptional cases where the root v is equal to o, to $(\pi(o), \ell(o)+1)$, or horizontally connected to the latter, one shows with similar arguments that if R_1, R_2 are two singular rays with the *same* exceptional root v, then the roots q_1, q_2 of the associated singular rays in $\mathrm{Hyp}(Z, \rho')$ are uniformly close to each other. Since there are only 3 types of exceptional roots, it follows that the distance between the image $F(v)$ of the root and the root q of the ρ'-ray associated to z is uniformly bounded, $|F(v)q| \doteq 0$. □

It follows readily that a roughly isometric map between geodesic spaces which induces a surjective boundary map is a rough isometry. The only thing left to show in the proof of Theorem 7, then, is that $\partial_\infty F = id_Z$. That a sequence converging to ω is mapped to $\infty \in (Z, \rho')$ is clear by definition of F. If $\{v_i\}$ is a sequence converging to infinity, say $\{v_i\} \in z, z \neq \omega$, we may suppose by [3] Lemma 6.3.2 that the v_i form a radial geodesic in $\mathrm{Hyp}(Z, \rho)$. Since F is a rough isometry, $\{F(v_i)\}$ converges to a point $z' \in (Z, \rho')$. But $F(v_i)$ is the smallest ρ'-ball containing $B_\rho(v_i)$, which contains z. Since $\rho(\pi(v_i), \pi(F(v_i))) \overset{i\to\infty}{\to} 0$ (the levels of $F(v_i)$ go to infinity), we have $\rho'(\pi(F(v_i)), z) \overset{i\to\infty}{\to} 0$, i.e. $\partial_\infty F(z) = z\ \forall z \in Z$. This completes the proof of Theorem 7.

7 Extension for P-QM maps

In this section we prove

Theorem 8 *Let $f : (Z, \rho) \to (Z', \rho')$ a power quasimoebius homeomorphism between complete quasimetric spaces. Then there exists a power quasi-isometry $F : \mathrm{Hyp}(Z) \to \mathrm{Hyp}(Z')$ with $\partial_\infty F = f$.*

The idea of the proof is to factor f as a composition of inversions and a P-QS map. We follow 3.15 of [10], where this factorization is explained in the metric setting.

Lemma 11 (Cf. [10], Theorem 2.1) *Let (X, ρ), (Y, ρ') be bounded quasimetric spaces and $f : X \to Y$ be θ-quasimoebius. Let $z_1, z_2, z_3 \in X$ and $\lambda > 0$ be such that $\rho(z_i, z_j) \geq$*

$\operatorname{diam}(X)/\lambda$ *and* $\rho'(f(x_i), f(x_j)) \geq \operatorname{diam}(Y)/\lambda$ *when* $i \neq j$*. Then there is a homeomorphism* $\mu : [0, \infty) \to [0, \infty)$*, depending only on* θ *and* λ *and the quasimetric constant* K *of* X*, such that*

$$\rho'(f(x), f(y)) \leq \operatorname{diam}(Y)\mu(\rho(x, y)/\operatorname{diam}(X)).$$

Moreover, if θ *is of power type, then* μ *can also be taken of power type.*

Proof In analogy to the proof of [10], Theorem 2.1, consider the cases

I. $\rho(x, z_1) < 1/K$ and $\rho(x, y) < 1/K^2$,
II. $\rho(x, y) \geq 1/K^2$,
III. $\rho(x, z_1) \geq 1/K$,

and follow the same arguments as in that proof, replacing any occurrence of the usual triangle inequality by the quasimetric version. Although not mentioned in [10], the fact that μ inherits power type is implied by the proof. □

Lemma 12 (Cf. [10], Theorem 3.12) *Suppose* $f : X \to Y$ *is a QM map between bounded quasimetric spaces. Then* f *is QS. If* f *is P-QM, then* f *is P-QS.*

Proof Also here the proof of [10] can be "quasified". Set $r_0 := \mu^{-1}(\mu^{-1}(1/K^2))$ and $r_1 := \min\{1/(K^2 t), r_0/(Kt), r_0/K\}$. Then consider the cases

I. $r \geq r_1$,
II. $r < r_1$ and $\rho'(f(x), f(z_1)) \geq \mu^{-1}(1/K^2)$,
III. $r < r_1$ and $\rho'(f(x), f(z_1)) < \mu^{-1}(1/K^2)$,

and follow analogous arguments to [10]. Careful inspection of that proof also yields the inheritance of power type. □

Lemma 13 below is the quasimetric analog of [10], Theorem 1.10.

Lemma 13 (Compare [10], Theorem 1.10) *Every (complete) quasimetric space* (Z, ρ) *is Moebius equivalent to a (complete) bounded quasimetric space.*

Proof Fix a $z_0 \in Z$, consider the set $Y = Z \cup \{\xi\}$, and equip it with the quasimetric $\widetilde{\rho}$ defined as $\widetilde{\rho}|_{Z\times Z} = \rho$ and $\widetilde{\rho}(\xi, z) = 1 + \rho(z_0, z)$. Then the canonical embedding $\iota : Z \hookrightarrow Z \cup \{\xi\}$ is an isometry. Invert $\widetilde{\rho}$ in ξ. □

We now have all the tools to prove the theorem.

Proof (Proof of Theorem 8) Let $\iota_i : (Z_i, \rho_i) \hookrightarrow Y_i$, $i = 1, 2$, be the embeddings as in the proof above. Let $z_i \in Z_i$ be fixed and denote by $u_i : Y_i \to Y_i$ the inversion in z_i as in the proof above. Then $v_i := u_i \circ \iota_i$ are Moebius homeomorphisms from (Z_i, ρ_i) onto their bounded images in Y_i.

Then $g := (u_2 \circ \iota_2) \circ f \circ (u_1 \circ \iota_1)|^{-1}_{u_1\circ\iota_1(Z_1)}$ is a PQ-Moebius homeomorphism between two bounded quasimetric spaces, thus it is PQ-symmetric by Lemma 12.

Thus f decomposes as $f = (u_2 \circ \iota_2)^{-1} \circ g \circ (u_1 \circ \iota_1)$. The claim follows with Theorems 7 and 5. Note that $\partial_\infty(F \circ G) = \partial_\infty F \circ \partial_\infty G$, purely by definition of the boundary maps. □

8 Main theorems

We recall the theorem stated in the introduction.

Theorem 9 *Let X, X' hyperbolic metric spaces with X visual and X' roughly geodesic.*

If $f : \partial_\infty^{a,o} X \to \partial_\infty^{a,o'} X'$ is a bilipschitz map then there exists a rough isometric map $F : X \to X'$ such that $\partial_\infty F = f$.

If $f : \partial_\infty^{o} X \to \partial_\infty^{o'} X'$ is a power quasisymmetric map then there exists a power quasi-isometric map $F : X \to X'$ such that $\partial_\infty F = f$.

If $f : \partial_\infty X \to \partial_\infty X'$ is a power quasimoebius map then there exists a power quasi-isometric map $F : X \to X'$ such that $\partial_\infty F = f$.

Proof Lemma 14 below reduces this to the case of surjective boundary maps. The theorem then follows from the extension theorems for P-QS and P-QM maps, Theorems 5 and 8, respectively, and the fact that by Corollary 1, (i) any visual hyperbolic space embeds into a hyperbolic approximation of its boundary and (ii), a hyperbolic approximation embeds into any geodesic hyperbolic space with the same boundary. □

Lemma 14 *Let (Z, ρ) be a complete quasi-metric space and $A \subset Z$ such that $(A, \rho|_A)$ is complete. Then $\mathrm{Hyp}_r(A)$ embeds roughly isometrically into $\mathrm{Hyp}_r(Z)$.*

Proof Let $\mathcal{V}_A$ be a vertex system for A and $\mathrm{Hyp}(A)$ its corresponding graph. By a Zorn-type argument we can extend $\mathcal{V}_A$ to a vertex system $\mathcal{V}$ for Z. Let $\mathrm{Hyp}(Z)$ be the resulting graph. It is now obvious that the canonical inclusion $\mathrm{Hyp}(A) \hookrightarrow \mathrm{Hyp}(Z)$ is roughly isometric. Since the approximation is independent of the choice of vertex system, the claim follows. □

As a special case of Theorem 9 we have

Theorem 10 *Let X, X' be visual roughly geodesic hyperbolic metric spaces. The following are mutually equivalent.*

- *(I) X and X' are roughly isometric.*
- *(II) There is a map $F : X \to X'$ and a $D \geq 0$ such that for all quadruples $Q \subset X$*

$$\mathrm{cd}(Q) - D \leq \mathrm{cd}(F(Q)) \leq \mathrm{cd}(Q) + D.$$

- *(III) For any $a > 1$ there is a bilipschitz-quasimoebius homeomorphism $f : \partial_\infty^a X \to \partial_\infty^a X'$.*

Also the following are mutually equivalent.

- *(i) X and X' are quasi-isometric.*
- *(ii) X and X' are power quasi-isometric.*
- *(iii) For any $a, a' > 1$, $\partial_\infty^a X$ is power quasimoebius equivalent to $\partial_\infty^{a'} X'$.*

Proof As mentioned in the introduction, the implications $(II) \Rightarrow (I)$, $(I) \Rightarrow (II)$ and $(ii) \Rightarrow (i)$ are all trivial.

$(I) \Rightarrow (III)$: It is clear that if X, X' are roughly isometric, then $\partial_\infty^{a;o} X$, $\partial_\infty^{a;o'} X'$ are bilipschitz equivalent for any $o \in X$, $o' \in X'$. Also, a bilipschitz map is obviously bilipschitz-quasimoebius. It remains to show that $\partial_\infty^{a,o} X$ and $\partial_\infty^{a,b} X$, with $b \in B(\omega)$ for some $\omega \in \partial_\infty X$, are bilipschitz-quasimoebius equivalent. But if we take the distinguished

Busemann function $b_{\omega,o}(x) := (\omega|o)_x - (\omega|x)_o$, it by definition induces the inverted quasimetric $\rho'(\cdot,\cdot) = a^{-(\cdot|\cdot)_{b_{\omega,o}}}$ to $\rho(\cdot,\cdot) = a^{-(\cdot|\cdot)_o}$ on $\partial_\infty X$,

$$\rho'(\xi,\eta) = \frac{\rho(\xi,\eta)}{\rho(\xi,\omega)\rho(\eta,\omega)},$$

so that $\partial_\infty^{a,o} X$ and $\partial_\infty^{a,b_{\omega,o}} X$ are Moebius-equivalent (no quasi). Now, by definition (cf. [3], §3.1), any $b \in \mathcal{B}(\omega)$ satisfies $b \doteq b_{\omega,o} - C$, for some C, and thus $\partial_\infty^{a,b} X$ and $\partial_\infty^{a,b_{\omega,o}} X$ are bilipschitz-quasimoebius equivalent.

(*III*) $\Rightarrow$ (*I*): By Theorem 7 and Remark 4, we may pre- and post-compose with inversions if necessary to reduce this to the bounded case $\partial_\infty^{a,o} X$, $\partial_\infty^{a,o'} X'$. By Lemma 15, f is bilipschitz. The claim now follows from the bilipschitz extension Theorem [3], Theorem 7.1.2 (cf. Theorem 4 above).

(i) $\Rightarrow$ (ii): This is Theorem 4.4.1 of [3].
(i) $\Rightarrow$ (iii): This is Proposition 5.2.10 of [3].
(iii) $\Rightarrow$ (i): This follows from the third statement of Theorem 9 and the fact, due essentially to the stability of quasi-geodesics, that a quasi-isometric map between visual (roughly) geodesic spaces which induces a bijective boundary map is necessarily a quasi-isometry (cf. [3], Lemma 7.3.12). □

Lemma 15 *If $f : Z \to Z'$ is a bilipschitz-QS map between quasimetric spaces, then f is bilipschitz. Consequently, if Z, Z' are bounded and f is bilipschitz-QM, then f is bilipschitz.*

Proof Let f be bilipschitz-QS, i.e. η-QS with $\eta(t) = \mu t$ for some constant μ. Fix $a, b \in Z, a \neq b$, and set $\Delta := |a'b'|/|ab|$, where $'$ denotes images under f. For $x \neq y$ in Z, $y \neq a$, $x \notin \{a, b\}$, we have

$$\frac{|x'y'|}{|xy|} \asymp_\mu \frac{|x'a'|}{|xa|} \asymp_\mu \frac{|a'b'|}{|ab|} = \Delta,$$

whence $|x'y'| \asymp_{\mu^2\Delta} |xy|$. The exceptional cases $y = a$, $x = a$ or $x = b$ are treated the same way.

For the second statement we remark as an addendum to Lemma 12, that one sees by going through the proof of [10], Theorem 3.12, that a bilipschitz-QM map between bounded quasimetric spaces is bilipschitz-QS. □

We give a couple of trivial examples that illustrate why we require the spaces to be visual and (roughly) geodesic. Indeed, given any visual hyperbolic metric space we can "stick on branches" of increasing length to destroy visibility and have no hope of recovering the space from its boundary. Likewise, the (rough) geodesic property guarantees that there are no holes (only holes of uniformly bounded size) in the space. It is clear that in some form or another, such a property is necessary to recover the space from its boundary.

Example 1 Not visual: The tree consisting of $\mathbb{R}_{\geq 0}$ with branches of length n branching off from the integer n is hyperbolic and has the same boundary as $\mathbb{R}_{\geq 0}$, but is not quasi-isometric to $\mathbb{R}_{\geq 0}$.

Not roughly geodesic: $\{2^n | n \in \mathbb{N}\}$ is obviously visual and hyperbolic, but not quasi-isometric to $\mathbb{R}_{\geq 0}$.

We now comment on how previous results fit into this context of visual spaces and "power maps."

In [1], Theorem 7.4 together with Theorem 8.2, Bonk and Schramm prove the bilipschitz- and P-QS extension theorems. The only difference is that our corresponding Theorems 4 and 5 are stated for arbitrary quasimetric boundary spaces, while Bonk and Schramm work under the assumption that the boundaries are equipped with visual metrics.

Buyalo and Schroeder added an extension theorem for quasimoebius maps, and did not require the boundary maps to be of power type.

Corollary 4 (Compare [3], Theorems 7.2.1, 7.3.1) *Let (Z, ρ), (Z', ρ') be uniformly perfect quasimetric spaces. Then any QS (QM)-map $f : Z \to Z'$ induces a quasi-isometric map $F : \mathrm{Hyp} Z \to \mathrm{Hyp} Z'$ with $\partial_\infty F = f$.*

Proof Any QS-map with uniformly perfect domain is P-QS, cf. [9], Theorem 3.10 (the proof works also in the quasimetric setting). Since inversions are Moebius (in the strict sense) and uniform perfection is invariant under inversions (cf. [5]), the same holds for QM maps. The corollary thus follows from Theorem 9. □

Much earlier, Paulin had already proved the following special case.

Corollary 5 (Compare [6], Théorème 1.4) *Let X, Y be two proper geodesic hyperbolic spaces with cobounded isometry groups. Then if $g : \partial_\infty X \to \partial_\infty Y$ is a quasimoebius homeomorphism, there is a quasi-isometry $G : X \to Y$ such that $\partial_\infty G = g$.*

In particular, two hyperbolic groups are quasi-isometric if and only if their boundaries are quasimoebius equivalent.

Proof By [3], Theorem 2.3.2, the boundary of a cobounded proper geodesic hyperbolic space is locally self-similar and in particular uniformly perfect. Thus g is *power* quasimoebius. The Cayley graph of a hyperbolic group is visual. The claim follows. □

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